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The response of Helmholtz resonators to external excitation. Part 1. Single resonators

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The response of single two- and three-dimensional Helmholtz resonators subject to external excitation by a plane acoustic wave is studied. The inviscid linearized problem is solved by the matched-asymptotic-expansion technique in the low-frequency limit, i.e. when the characteristic neck dimension is small compared with the acoustic wavelength. The scale of the cavity is chosen such as to tune the system to the lowest Helmholtz mode. The results are compared with the classical lumped-element model, and refined expressions are derived for the effective mass (added length) and stiffness (effective cavity volume) of the resonator.

1. Introduction

In the present paper we analyse the response of Helmholtz resonators, which are widely used as sound-management devices and which consist of a resonator cavity with, in our case, an opening or neck in an infinite rigid baffle (cf. figures 1 and 6). In practice and in our study the resonator is excited by a plane wave incident upon the baffle, and the purpose of the models and theories is to predict the response of the resonator in terms of acoustic impedance, defined customarily as the ratio of incident acoustic pressure to average normal velocity at the outside neck plane. The result is mostly used to determine the resonance frequency of the device where the absolute value of the (complex) impedance is a minimum. The basic physical model of the Helmholtz resonator includes an air spring (the cavity), a mass (the air mass in the neck plus some added mass adjacent to the neck) and a damper, corresponding to three terms in the impedance.

These three terms have in turn been modelled with various degrees of sophistication. For cavities small compared with the acoustic wavelength the spring constant is determined by relating the volume flow through the neck and the uniform pressure in the cavity. The more difficult problem of the added mass or added length l' of the neck has been addressed by Rayleigh (1945), who gave lower and upper bounds for l' by considering constant pressure and constant velocity respectively in the neck exit plane. A common practice to improve the upper bound of l' is to assume a suitably parametrized family of velocity profiles at the orifice and to invoke the 'principle of minimum kinetic energy' (see Rayleigh 1945, Appendix A). The losses finally are generally thought to consist of viscous losses within the Stokes boundary layer in the neck, and of radiation losses.

An extensive review of the lumped-elements approach (the spring–mass damper model) can be found in Ingard (1953). Addressing the problem of harbour resonance, which is directly related to the resonance of a two-dimensional Helmholtz resonator,

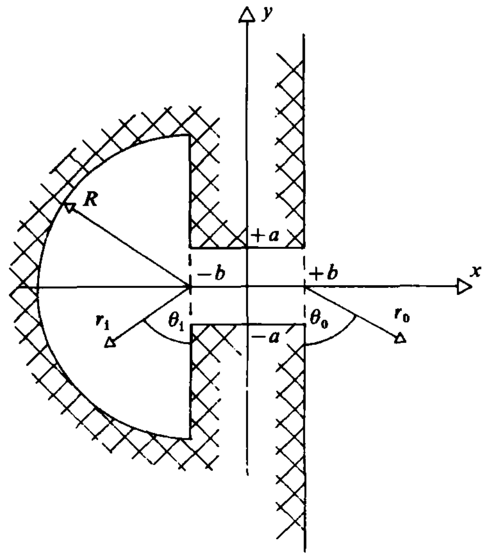

FIGURE 1. The geometry of the 2-dimensional Helmholtz resonator.

Carrier, Shaw & Miyata (1971) have suggested a refined approach in which the external wave field, the mouth and the harbour basin (the cavity) are treated separately: the solutions of the (linear) wave equation are sought in each region and patched together using continuity of force (pressure) and flux (average velocity). Miles (1971) and Miles & Lee (1975) combine these ideas with the electrical analogy of the lumped-element model. To arrive at an estimate of the harbour impedance, they use a variational approach related to Rayleigh's minimum kinetic-energy principle.

A common characteristic of all the treatments mentioned above is the lack of error estimates in terms of the ratio of, say, neck width to acoustic wavelength and the lack of a formal procedure to improve their accuracy.

In this paper we will remedy these deficiencies, at least for simple geometries, by seeking solutions of the linearized inviscid basic equations in terms of (singular) low-frequency expansions in the exterior, neck and cavity and by asymptotically matching them. This process, which can formally be extended to any order in the expansion parameter (which in our case is the product of acoustic wavenumber and characteristic neck dimension) has already been used successfully in related problems by Lesser & Lewis (1972*a*, *b*). For a general overview see also Lesser & Crighton (1976).

In the following the analysis will be developed in detail for a two-dimensional resonator of arbitrary (although small compared with the acoustic wavelength) neck length. For this case the acoustic energy balance, and specifically the redistribution of energy by the resonator, will be discussed. A special three-dimensional resonator with a neck of zero length will also be considered as an independent check of the solution method, and the results will be discussed along the way.

2. The two-dimensional resonator

A single resonator in an infinite baffle, as shown in figure 1, is considered. The semicylindrical shape of the cavity has been chosen for mathematical simplicity. In the following the excitation is achieved by a plane sound wave incident upon the baffle, and the resonance is calculated by a linearized inviscid analysis. The result will be presented in the form of an acoustic impedance, defined as the ratio of average pressure (or incident pressure) and average normal velocity at the outside end of the orifice ($x = b$, $-a \leqslant y \leqslant a$).

First the acoustic disturbance quantities (subscript a) are nondimensionalized with mean quantities (subscript 0) in the following way:

$$\left.\begin{aligned} \rho = \frac{\rho_a}{\rho_0}, \quad p = \frac{p_a}{\rho_0 c_0^2}, \quad \boldsymbol{u} = \frac{\boldsymbol{u}_a}{c_0}, \\ \hat{\boldsymbol{x}} = \hat{k}\boldsymbol{x}; \quad \hat{t} = \omega t \quad \left(\hat{k} = \frac{\omega}{c_0}\right). \end{aligned}\right\} \tag{2.1}$$

The coordinates are scaled with the acoustic wavelength to adapt the formulation to the region $x \geqslant b$ where acoustic waves are propagating. Assuming harmonic forcing proportional to $\exp[-\mathrm{i}\omega t] \equiv \exp[-\mathrm{i}\hat{t}]$, the linearized equations of energy, together with the equation of state for a perfect gas, continuity and momentum take the form

$$\rho = p, \quad \hat{\nabla}\cdot\boldsymbol{u} = \mathrm{i}\rho, \quad \hat{\nabla} p = \mathrm{i}\boldsymbol{u}. \tag{2.2}$$

Combining these equations leads to the Helmholtz equation

$$\hat{\nabla}^2 p + p = 0. \tag{2.3}$$

Next we turn to the solution of these equations in the Helmholtz region, the neck region and the cavity. After this the solutions in the different regions are matched in §2.4 and the acoustic impedance is derived subsequently.

2.1. *The Helmholtz region*

In this region $\hat{x} \geqslant \hat{b}$, the Helmholtz equation has to be solved subject to the boundary conditions

$$\left.\begin{aligned} \frac{\partial \hat{p}}{\partial \hat{x}} = 0 \quad \text{for } \hat{x} = \hat{b} \text{ and } |\hat{y}| \geqslant \hat{a}, \\ \text{and a radiation condition (except for the incident wave) as } \hat{r}_0 \to \infty. \end{aligned}\right\} \tag{2.4}$$

The hat on the acoustic quantities indicates a solution pertinent to the Helmholtz region. For the following we will consider an excitation by a plane wave coming from the fourth quadrant with the angle φ enclosed between its wavevector and the x-axis. The incident wave $\frac{1}{2}\hat{p}_i \exp[-\mathrm{i}(\hat{x}-\hat{b})\cos\varphi + \mathrm{i}\hat{y}\sin\varphi]$ can immediately be combined with its reflection from the infinite baffle to give

$$\begin{aligned} \hat{p} &= \hat{p}_i \cos[(\hat{x}-\hat{b})\cos\varphi]\,\mathrm{e}^{\mathrm{i}\hat{y}\sin\varphi} \\ &\approx \hat{p}_i\{1 - \mathrm{i}\hat{r}_0\cos\theta_0\sin\varphi - \tfrac{1}{4}\hat{r}_0^2(1-\cos 2\theta_0\cos 2\varphi) + O(\hat{r}_0^3)\} \quad \text{as } \hat{r}_0 \to 0. \end{aligned} \tag{2.5}$$

The behaviour of the solution close to the resonator mouth is recorded for later use. All other waves in this region must originate at the resonator mouth. Hence, for an opening small compared with the acoustic wavelength, they must be of the form

$$\hat{p}_n = \tfrac{1}{2}\mathrm{i}\pi \hat{A}_n H_n^{(1)}[\hat{r}_0]\cos[n\theta_0]. \tag{2.6}$$

The behaviour of $\hat{p}_0$ and $\hat{p}_1$ for small $\hat{r}_0$ is again listed below for convenience:

$$\left.\begin{aligned} \hat{p}_0 &\approx -\hat{A}_0\{\ln\tfrac{1}{2}\hat{r}_0 + \gamma - \tfrac{1}{2}\mathrm{i}\pi - \tfrac{1}{4}\hat{r}_0^2(\ln\tfrac{1}{2}\hat{r}_0 + \gamma - 1 - \tfrac{1}{2}\mathrm{i}\pi) + O(\hat{r}_0^4\ln\hat{r}_0)\}, \\ \hat{p}_1 &\approx \hat{A}_1\cos\theta_0\left\{\frac{1}{\hat{r}_0} - \tfrac{1}{2}\hat{r}_0(\ln\tfrac{1}{2}\hat{r}_0 + \gamma - \tfrac{1}{2} - \tfrac{1}{2}\mathrm{i}\pi) + O(\hat{r}_0^3\ln\hat{r}_0)\right\}, \end{aligned}\right\} \tag{2.7}$$

where $\gamma = 0.5772\ldots$ is the Euler constant.

2.2. *The neck region*

The lengthscale in the neck region is taken to be the half-width a of the orifice, where a is assumed to be very much smaller than the acoustic wavelength. $\hat{a} = a\hat{k}$ will therefore play the role of a small expansion parameter in this problem,

$$\hat{a} = a\hat{k} \equiv \epsilon \ll 1, \tag{2.8}$$

and the coordinates in this region are magnified according to

$$\tilde{\boldsymbol{x}} = \hat{\boldsymbol{x}}/\epsilon. \tag{2.9}$$

The equations (2.2) and (2.3) in neck variables then take the form

$$\left.\begin{aligned} \tilde{\nabla}\cdot\tilde{\boldsymbol{u}} = \mathrm{i}\epsilon\tilde{p}, \quad \tilde{\nabla}\tilde{p} = \mathrm{i}\epsilon\tilde{\boldsymbol{u}}, \\ \tilde{\nabla}^2\tilde{p}+\epsilon^2\tilde{p} = 0. \end{aligned}\right\} \tag{2.10}$$

The last of the above equations shows that to leading order the velocity field in a neighbourhood of the neck is the same as for an ideal incompressible fluid. To take advantage of this fact, $\tilde{p}$ and $\tilde{\boldsymbol{u}}$ are expanded in powers of ϵ^2:

$$\left.\begin{aligned} \tilde{p}_n &= \tilde{p}_{n,0}+\epsilon^2\tilde{p}_{n,1}+O(\epsilon^4), \\ \tilde{\boldsymbol{u}}_n &= \epsilon^{-1}\tilde{\boldsymbol{u}}_{n,0}+\epsilon\tilde{\boldsymbol{u}}_{n,1}+O(\epsilon^3) \end{aligned}\right\} \tag{2.11}$$

The first index n has been added at this point to indicate the leading behaviour of the solution away from the neck, i.e. for $\tilde{r}_0$ and $\tilde{r}_1 \rightarrow \infty$, in accordance with the index in (2.6). The solutions $n = 0, 1, 2$ as seen from far away therefore behave like a source, dipole and quadrupole respectively. Inserting (2.11) into (2.10) readily yields:

$$\left.\begin{aligned} \tilde{\nabla}^2\tilde{p}_{n,0} = 0, \quad \tilde{\nabla}^2\tilde{p}_{n,1} = -\tilde{p}_{n,0}, \quad \ldots, \\ \tilde{\nabla}\cdot\tilde{\boldsymbol{u}}_{n,0} = 0, \quad \tilde{\nabla}\cdot\tilde{\boldsymbol{u}}_{n,1} = \mathrm{i}\tilde{p}_{n,0}, \quad \ldots. \end{aligned}\right\} \tag{2.12}$$

We now turn our attention to the solution of Laplace's equation for $\tilde{p}_{n,0}$ in the neck geometry of figure 1, where the length of the neck is assumed to be of the same order as its width, i.e. $\tilde{b} \equiv b/a = O(1)$. As it will turn out, the solutions depend regularly on $\tilde{b}$ such that one can safely consider the cases $\tilde{b} \rightarrow 0$ (a slit in an infinitely thin baffle) and $\tilde{b} \gg 1$ as long as there is no wave propagation, i.e. provided $\hat{b} \ll 1$.

The solution is most conveniently obtained by conformal mapping. An appropriate mapping from the $z = \tilde{x}+\mathrm{i}\tilde{y}$ plane onto the lower w-plane is found in Davy (1944), and is given in terms of Jacobi elliptic functions and complete elliptic integrals (using standard notation) as follows:

$$\left.\begin{aligned} z+\tilde{b} &= \frac{\mathrm{i}}{D}\left\{\zeta k'^2 - 2\int_0^\zeta \mathrm{dn}^2\, u \,\mathrm{d}u - \frac{\mathrm{cn}\,\zeta\;\mathrm{dn}\,\zeta}{\mathrm{sn}\,\zeta}\right\}, \\ w &= (\mathrm{sn}\,\zeta)^{-1}, \quad D = 2E-k'^2 K. \end{aligned}\right\} \tag{2.13}$$

Bearing in mind that $\tilde{a} = 1$, the modulus k and complementary modulus $k' = (1-k^2)^{\frac{1}{2}}$ are defined by the equation

$$-\tilde{b} = (K'k'^2 - 2K' + 2E')(2D)^{-1}. \tag{2.14}$$

The points $|w| \rightarrow 0$ and $|w| \rightarrow \infty$ thereby correspond to $\tilde{r}_0 \rightarrow \infty$ (outside) and $\tilde{r}_1 \rightarrow \infty$ (inside). Furthermore, the vertices $z = -\tilde{b} \pm \mathrm{i}$ correspond to $w = \mp 1$.

Solutions $\tilde{p}_{n,0}$ are now obtained by arranging singularities in the w-plane, i.e. for $\tilde{p}_{0,0}$ we place a source at the origin $w = 0$. The complex potential F is thus given by

$$F_0 = \phi_0 + \mathrm{i}\psi_0 = \ln w, \tag{2.15}$$

where $\tilde{p}_{0,0}$ is proportional to ϕ_0. Likewise, to obtain $\tilde{p}_{1,0}$ we place a dipole at $w = \infty$ or $w = 0$:

$$F_1^{(\pm)} = \phi_1^{(\pm)} + \mathrm{i}\,\psi_1^{(\pm)} = w^{\pm 1}. \tag{2.16}$$

It is clear that no explicit expressions for the $\phi_n(\tilde{x}, \tilde{y})$ can be obtained in the whole domain, but, in order to perform the matching with the Helmholtz and cavity solutions (see §2.4), we only need their asymptotic expansions for $\tilde{r}_1 \to \infty$ ($|w| \to \infty$) and $\tilde{r}_0 \to \infty$ ($|w| \to 0$).

For this we have to establish the asymptotic connection between z and w in both cases: from (2.13) it follows that $|w| \to \infty$ corresponds to $|\zeta| \to 0$ such that the series expansions for the Jacobi elliptic functions can be used (see Gradshteyn & Ryzhik 1965, §8.14)

$$\left.\begin{aligned} & w = (\operatorname{sn}\zeta)^{-1} \approx \zeta^{-1}\{1 + \tfrac{1}{6}(1+k^2)\zeta^2 + \ldots\}, \\ & \operatorname{dn}^2\zeta \approx 1 - k^2\zeta^2 + \ldots, \\ & \operatorname{cn}\zeta\,\operatorname{dn}\zeta\,(\operatorname{sn}\zeta)^{-1} \approx \zeta^{-1}\{1 - \tfrac{1}{3}(1+k^2)\zeta^2 + \ldots\} \\ & \Rightarrow z + \tilde{b} \propto -\frac{\mathrm{i}}{D}\left\{w + \frac{1+k^2}{2w} + \frac{(1-k^2)^2}{24w^3} + O(|w|^{-5})\right\} \quad \text{as } |w| \to \infty. \end{aligned}\right\} \tag{2.17}$$

The other limit $|w| \to 0$ corresponds to $\zeta \to \mathrm{i}K'$. With $\zeta = \mathrm{i}K' + \zeta'$ one obtains

$$\left.\begin{aligned} & w \approx k\zeta', \\ & \int_0^\zeta \operatorname{dn}^2 u\,\mathrm{d}u \approx (\zeta')^{-1} + \frac{\mathrm{i}}{K}(EK' - \tfrac{1}{2}\pi) + O(|\zeta^1|) \\ & \operatorname{cn}\zeta\,\operatorname{dn}\zeta\,(\operatorname{sn}\zeta)^{-1} \approx -(\zeta')^{-1} \\ & \Rightarrow z - \tilde{b} \approx -\frac{\mathrm{i}k}{Dw} + O(|w|) \quad \text{as } |w| \to 0. \end{aligned}\right\} \tag{2.18}$$

For the source potential (2.15) we obtain with (2.17) two equations for ϕ_0 and ψ_0:

$$\left.\begin{aligned} D^2(\tilde{x} + \tilde{b})^2 &= \sin^2\psi_0\, \mathrm{e}^{2\phi_0}\{1 - (1+k^2)\,\mathrm{e}^{-2\phi_0} + \ldots\}, \\ D^2\tilde{y}^2 &= \cos^2\psi_0\, \mathrm{e}^{2\phi_0}\{1 + (1+k^2)\,\mathrm{e}^{-2\phi_0} + \ldots\}. \end{aligned}\right\} \tag{2.19}$$

With $\tilde{r}_1^2 = (\tilde{x} + \tilde{b})^2 + \tilde{y}^2$ we finally obtain

$$\phi_0 \propto \ln[D\tilde{r}_1] - \frac{(1+k^2)\cos 2\theta_1}{2D^2\tilde{r}_1^2} + O\left(\frac{\cos 4\theta_1}{\tilde{r}_1^4}\right), \quad (\tilde{r}_1 \to \infty). \tag{2.20}$$

The expansion of ϕ_0 for $\tilde{r}_0 \to \infty$ can be obtained from (2.18) or by noting that ϕ_0 has to be antisymmetric with respect to $\tilde{x} = 0$. From Davy (1944) it is known that the points $z = \pm\mathrm{i}$ correspond to $w = \pm k^{\frac{1}{2}}$; hence $\phi_0(\tilde{x} = 0) = \tfrac{1}{2}\ln k$ and

$$\phi_0(\tilde{r}_0, \theta_0) - \tfrac{1}{2}\ln k = -[\phi_0(\tilde{r}_1, \theta_1) - \tfrac{1}{2}\ln k]. \tag{2.21}$$

After noting that an arbitrary constant may be added to the solution, the pressure in the neck region due to a source at $|x| \to \infty$ is given asymptotically by

$$\left.\begin{aligned} \tilde{p}_{0,0} &\propto -\tilde{A}_0[\ln\tilde{r}_0 + \mathcal{D}] + \tilde{B}_0 \quad (\tilde{r}_0 \to \infty), \\ \tilde{p}_{0,0} &\propto +\tilde{A}_0[\ln\tilde{r}_1 + \mathcal{D}] + \tilde{B}_0 \quad (\tilde{r}_1 \to \infty), \end{aligned}\right\} \tag{2.22}$$

with

$$\mathcal{D} = \ln[Dk^{-\frac{1}{2}}].$$

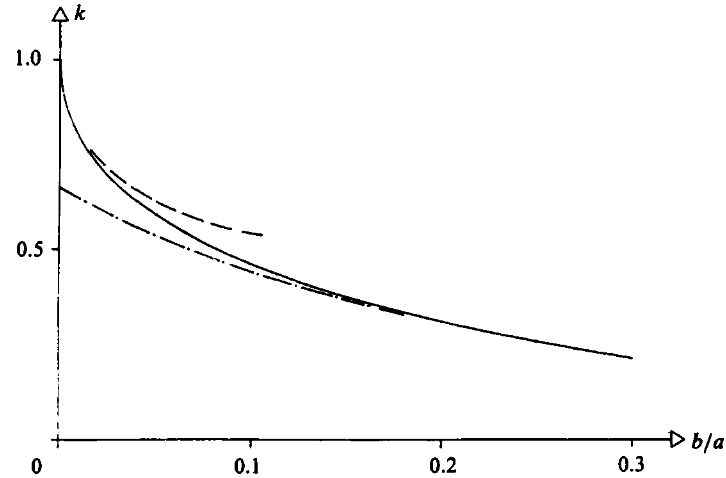

FIGURE 2. The modulus k of the elliptic functions in (2.13) versus neck length-to-width ratio b/a: ——, numerical solution of (2.14); — —, approximation (2.23); —·—, approximation (2.24).

In the following, expressions for D in the case of small and large $\tilde{b}$ are derived. Upon inspection of (2.14) we find that $\tilde{b} \ll 1$ corresponds to $k' \ll 1$, whereas $\tilde{b} \gg 1$ corresponds to $k \ll 1$. Using the expansion of the complete elliptic integrals (Gradshteyn & Ryzhik 1965, §8.11), we obtain

for $\tilde{b} \ll 1$

$$\left.\begin{aligned} D &\approx 2\left\{1-\frac{k'^2}{4}+\frac{k'^4}{16}\left(\ln\frac{4}{k'}-\frac{5}{4}\right)+\dots\right\} \\ k &\approx 1-4\left(\frac{\tilde{b}}{\pi}\right)^{\frac{1}{2}}+8\frac{\tilde{b}}{\pi}+\dots \\ \mathbb{D} &= \ln 2+\frac{\tilde{b}}{\pi}\left(1+\ln\frac{4\pi}{\tilde{b}}\right)+O(\tilde{b}^{\frac{3}{2}}), \end{aligned}\right\} \tag{2.23}$$

and for $\tilde{b} \gg 1$

$$\left.\begin{aligned} D &\approx \tfrac{1}{2}\pi\,\{1+\tfrac{1}{4}k^2+\tfrac{1}{64}k^4+\dots\}, \\ k &\approx 4\,\mathrm{e}^{-(\pi\tilde{b}+2)}\{1+12\,\mathrm{e}^{-2(\pi\tilde{b}+2)}+\dots\}, \\ \mathbb{D} &= \tfrac{1}{2}\pi\tilde{b}+1+\ln\tfrac{1}{4}\pi-2\,\mathrm{e}^{-2(\pi\tilde{b}+2)}+O(\mathrm{e}^{-4(\pi\tilde{b}+2)}). \end{aligned}\right\} \tag{2.24}$$

In figure 2 the approximations (2.23) and (2.24) for k are compared with a numerical solution of (2.14). It appears that for $\tilde{b} = b/a > 0.15$, (2.24) constitutes a very good approximation, Also of interest for the impedance calculation in §2.4 are the average (averaged over $\tilde{y}$) velocity and pressure at the mouth $\tilde{x} = \tilde{b}$ associated with the solution $\tilde{p}_{0,0}$. Using (2.12) and the divergence theorem on the area S_1 (see figure 3) readily yields the average $\tilde{x}$-velocity

$$\bar{\tilde{u}}_{0,0} = \tfrac{1}{2}\mathrm{i}\pi\,\tilde{A}_0. \tag{2.25}$$

The $\tilde{y}$-component of Stokes' theorem on the contour ∂S_2 (figure 3) together with the symmetry condition (2.21) and the result (2.25) gives the average pressure $\bar{\tilde{p}}_{0,0}$ at $\tilde{x} = \tilde{b}$:

$$\bar{\tilde{p}}_{0,0} = -\tfrac{1}{2}\pi\tilde{b}\tilde{A}_0+\tilde{B}_0. \tag{2.26}$$

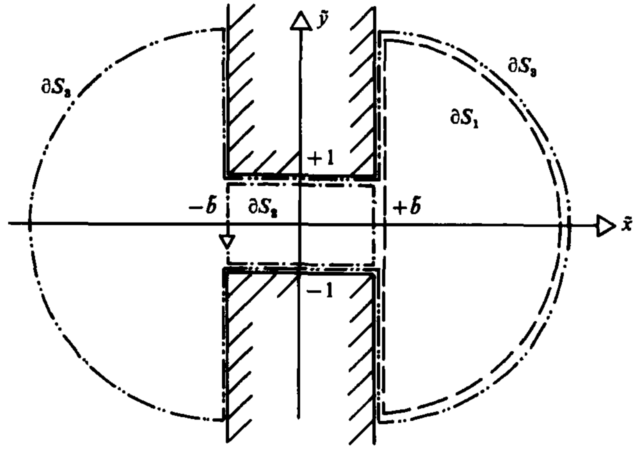

FIGURE 3. Integration paths and domains for the application of the divergence and Stokes' theorems.

We next turn to the particular solutions $\tilde{\boldsymbol{u}}_{0,1}$ and $\tilde{p}_{0,1}$. As $\tilde{p}_{0,0}-\tilde{B}_0$ is antisymmetric with respect to $\tilde{x}=0$, $\tilde{\boldsymbol{u}}_{0,1}$ associated with it is symmetric. The asymptotic behaviour of a particular solution $\tilde{v}_{0,1}$, the radial velocity component is thus given by

$$\left.\begin{aligned} \tilde{v}_{0,1} &\propto -\mathrm{i}\tilde{A}_0\frac{\tilde{r}_0}{2}\{\ln\tilde{r}_0+D-\tfrac{1}{2}\}+\mathrm{i}\tilde{B}_0\frac{\tilde{r}_0}{2} \quad (\tilde{r}_0\to\infty),\\ \tilde{v}_{0,1} &\propto \mathrm{i}\tilde{A}_0\frac{\tilde{r}_\mathrm{i}}{2}\{\ln\tilde{r}_\mathrm{i}+D-\tfrac{1}{2}\}+\mathrm{i}\tilde{B}_0\left\{\frac{\tilde{r}_\mathrm{i}}{2}+\frac{c}{\tilde{r}_\mathrm{i}}\right\} \quad (\tilde{r}_\mathrm{i}\to\infty).\end{aligned}\right\} \tag{2.27}$$

Application of the divergence theorem on S_1 (figure 3) yields the average velocity $\bar{\tilde{u}}_{0,1}$ at $\tilde{x}=\tilde{b}$ with an unknown coefficient c_u multiplying $\tilde{A}_0$, while the same theorem on S_3 yields the constant $c=4\tilde{b}/\pi$ in (2.27):

$$\bar{\tilde{u}}_{0,1}=c_u\tilde{A}_0+0\times\tilde{B}_0. \tag{2.28}$$

Integrating (2.27) once, one obtains the asymptotic representation of the particular solution $\tilde{p}_{0,1}$:

$$\left.\begin{aligned} \tilde{p}_{0,1} &\propto \tilde{A}_0\frac{\tilde{r}_0^2}{4}\{\ln\tilde{r}_0+D-1\}-\tilde{B}_0\frac{\tilde{r}_0^2}{4} \quad (\tilde{r}_0\to\infty),\\ \tilde{p}_{0,1} &\propto -\tilde{A}_0\frac{\tilde{r}_\mathrm{i}^2}{4}\{\ln\tilde{r}_\mathrm{i}+D-1\}-\tilde{B}_0\left\{\frac{\tilde{r}_\mathrm{i}^2}{4}+\frac{4\tilde{b}}{\pi}\ln\tilde{r}_\mathrm{i}\right\} \quad (\tilde{r}_\mathrm{i}\to\infty).\end{aligned}\right\} \tag{2.29}$$

Similarly we obtain for the dipole (2.16) the results listed below. The use of the divergence theorem again leads to the leading-order average velocity at the mouth $\tilde{x}=\tilde{b}$:

$$\left.\begin{aligned} &\tilde{p}_{1,0}^{(\pm)} \propto \cos\theta_0\tilde{A}_1^{(\pm)}\left(\frac{D\tilde{r}_0}{k}\right)^{\mp1},\\ &\tilde{p}_{1,1}^{(+)} \propto -\cos\theta_0\tilde{A}_1^{(+)}\left(\frac{k}{2D}\right)\tilde{r}_0\ln\tilde{r}_0 \quad (\tilde{r}_0\to\infty),\\ &\tilde{p}_{1,0}^{(\pm)} \propto \cos\theta_\mathrm{i}\tilde{A}_1^{(\pm)}(D\tilde{r}_\mathrm{i})^{\pm1},\\ &\tilde{p}_{1,1}^{(-)} \propto -\cos\theta_\mathrm{i}\tilde{A}_1^{(-)}\frac{1}{2D}\tilde{r}_\mathrm{i}\ln\tilde{r}_\mathrm{i} \quad (\tilde{r}_\mathrm{i}\to\infty),\\ &\bar{\tilde{u}}_{1,0}^{(\pm)}=0.\end{aligned}\right\} \tag{2.30}$$

From the expressions (2.5) and (2.20) it is clear that quadrupoles and higher singularities have to be considered, but in the following we will restrict ourselves to matching radially symmetric terms (source terms) and terms proportional to $\cos\theta$ (dipole terms), as the procedure can easily be extended to include quadrupole and higher terms. With this we can now turn to the solution of the basic equation in the cavity.

2.3. *The cavity*

For mathematical simplicity a semicylindrical shape of the cavity has been chosen. The obvious lengthscale in this region is therefore the cavity radius R, which has to be specified in relation to the acoustic wavelength and the neck width.

In the following we will concentrate on the basic Helmholtz resonance, where the pressure in the cavity is constant to leading order. In this regime the results are expected to apply approximately to cavity shapes other than semicylindrical, provided the frequency is below the lowest cavity resonance; the correspondence is simply established by a volume-preserving transformation of the cavity into a semicylindrical shape. For this basic resonance we therefore require $\hbar R < 1$. For frequencies close to and above resonance of the cavity *alone*, which have been considered for instance by Carrier *et. al* (1971) and Miles (1971), the response depends strongly on the exact cavity shape. On the other side we also specify a lower bound for $\hbar R$ by requiring that R be much larger than the neck width. Therefore, simple quarter-wave tubes or similar 'wide-mouth' devices will not be considered. In summary we have to rescale the coordinates in the cavity according to

$$\bar{x} = \frac{\hat{x}}{\delta}, \quad \text{with } O(\epsilon) < \delta < O(1). \tag{2.31}$$

In order to make a rational choice for $\delta(\epsilon)$ in the range specified above, we note that Helmholtz resonators are only of practical interest around resonance. We therefore consider the thought experiment where the forcing frequency is kept constant while the neck dimensions are continuously decreased to zero (corresponding to $\epsilon \rightarrow 0$) and require that during this process the resonator stay tuned. In other words we require that the properly scaled radius $\bar{R} = \hat{R}/\delta$ corresponding to resonance be of order unity in the limit $\epsilon \rightarrow 0$. This requirement now enables us to determine $\delta(\epsilon)$ by taking recourse to the most basic spring-mass model. We recall that resonance is characterized by a minimum of the impedance, which is attained when the spring and the mass terms balance. Their ratio therefore has to be of order unity. Using dimensional quantities, where l', S and V are added length, mouth area and cavity volume respectively, we demand that

$$\hbar^2 l' V (S)^{-1} = O(1). \tag{2.32}$$

The reason for using the added length rather than the actual length $2b$ is that we have set $b = O(a)$ whereas l' is $O(a \ln[1/\epsilon])$ and therefore dominant (see e.g. Rayleigh 1945). Using $\hbar^2\, V$ proportional to $\hat{R}^2 = \delta^2\,\bar{R}^2$ and S proportional to a, one arrives at

$$\delta = (\ln[1/\epsilon])^{-\frac{1}{2}}. \tag{2.33}$$

The square of this parameter, in the limit $\epsilon \rightarrow 0$, is related to Miles' (1971) δ (his equation (1.4)), which in our notation is $(\ln \hat{R})^{-1}$. This will become clear from the resonance condition (2.56). This choice for δ is very close to unity; in fact, according to the block-matching idea of Lesser & Crighton (1976), the $O(\delta)$ quantities belong to the same 'block' as the $O(1)$ quantities. Therefore it is not advisable to solve the Helmholtz equation iteratively as in the neck region. The resulting series in powers of δ^2 would converge very slowly indeed. Instead, taking advantage of the simple cavity geometry, we choose to solve the complete Helmholtz equation (2.3) directly, subject to the boundary conditions

$$\left.\begin{aligned} \frac{\partial \bar{p}}{\partial \theta_i} &= 0 \quad \text{on } \theta_i = 0, \pi, \\ \frac{\partial \bar{p}}{\partial \bar{r}_i} &= 0 \quad \text{on } \bar{r}_i = \bar{R}. \end{aligned}\right\} \tag{2.34}$$

The solution is easily found in terms of the Bessel functions J_n and Neumann functions Y_n (sometimes also denoted by N_n). For the source ($\bar{p}_0$) and dipole ($\bar{p}_1$) solutions one obtains

$$\left.\begin{aligned} \bar{p}_0 &= \bar{A}_0\{J_0(\hat{r}_i) + \tfrac{1}{4}\delta^2\pi\bar{R}^2\mu_0(\delta\bar{R})\, Y_0(\hat{r}_i)\}, \\ \mu_0(z) &= -\frac{4J_0'(z)}{\pi z^2 Y_0'(z)}, \end{aligned}\right\} \tag{2.35}$$

$$\begin{aligned} \bar{p}_0 \approx \bar{A}_0\bigg\{1 &+ \frac{\mu_0\bar{R}^2}{2\ln[1/\epsilon]}(\ln\tfrac{1}{2}\hat{r}_i + \gamma) \\ &- \frac{\hat{r}_i^2}{4}\left[1 + \frac{\mu_0\bar{R}^2}{2\ln[1/\epsilon]}(\ln\tfrac{1}{2}\hat{r}_i + \gamma - 1)\right] + O(\hat{r}_i^4\ln\hat{r}_i)\bigg\} \quad \text{as } \hat{r}_i \to 0, \end{aligned} \tag{2.36}$$

$$\left.\begin{aligned} \bar{p}_1 &= \bar{A}_1\cos\theta_i\{J_1(\hat{r}_i) - \tfrac{1}{4}\delta^2\pi\bar{R}^2\mu_1(\delta\bar{R})\, Y_1(\hat{r}_i)\} \\ \mu_1(z) &= \frac{4J_1'(z)}{\pi z^2 Y_1'(z)}, \end{aligned}\right\} \tag{2.37}$$

$$\begin{aligned} \bar{p}_1 &\approx \bar{A}_1\cos\theta_i\left(\hat{r}_i\ln\frac{1}{\epsilon}\right)^{-1} \\ &\times\left\{\tfrac{1}{2}\mu_1\bar{R}^2 + \frac{\hat{r}_i^2}{4}\left[2\ln\frac{1}{\epsilon} - \mu_1\bar{R}^2(\ln\tfrac{1}{2}\hat{r}_i + \gamma - \tfrac{1}{2})\right] + O(\hat{r}_i^4\ln\hat{r}_i)\right\} \quad \text{as } \hat{r}_i \to 0. \end{aligned} \tag{2.38}$$

In the expressions above, the two functions μ_0 and μ_1, defined in terms of derivatives of Bessel functions, are both normalized to one in the limit of the argument $\delta\bar{R} \to 0$ and are plotted on figure 4. In the range $\delta\bar{R} < 1$ both functions are of order unity, but for larger arguments they have alternating poles and zeros. The first zero of μ_0, for instance, is found for $J_1(z) = 0$ or $z = 3.83$, which corresponds to the lowest radially symmetric resonance of the cavity alone. No resonances occur for arguments smaller than 1.84, the first zero of μ_1, at any multipole order, in confirmation of the previous discussion on the choice of δ. Again the expansions of $\bar{p}_0$ and $\bar{p}_1$ for small $\hat{r}_i = \delta\bar{r}$ are listed above for use in §2.4.

At this point the necessary solutions in all three regions have been derived and we can now proceed to their matching.

2.4. *Matching and results*

In the following the source and dipole solutions will be matched. As pointed out by Lesser & Crighton (1976), special care has to be exercised in the presence of logarithmic terms, which leads to their block-matching procedure. The present case would be potentially worse if the cavity solution were only known as a series expansion in δ^2 (in a more complicated geometry, for instance). In this case powers of both $\ln[1/\epsilon]$ and $\ln\ln[1/\epsilon]$ would appear. Fortunately these difficulties can be avoided here by simply not expanding the Bessel functions of the small argument

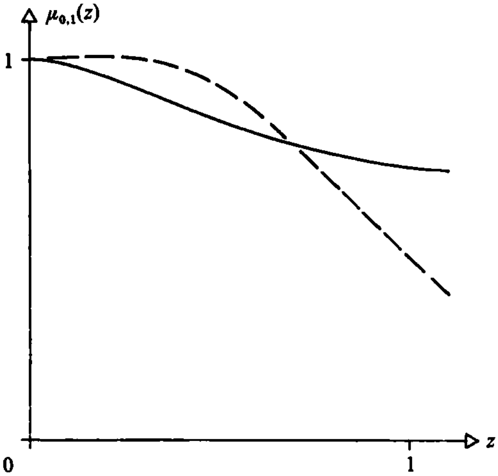

FIGURE 4. The two-dimensional volume-correction factors in the source and dipole solutions: ——, μ_0 defined by (2.35); – – –, μ_1 defined by (2.37).

$\hat{r}_1 = \delta\bar{r}$ in (2.35) and (2.37) and by treating μ_0 and μ_1 as $O(1)$ quantities. In the following we opt for the intermediate matching procedure of Cole (see Kevorkian & Cole 1981), which will yield the same result as the block-matching procedure.

To match the Helmholtz solution to the neck solution, we introduce the intermediate scaling

$$x^* = \frac{\hat{x}}{\epsilon^{\alpha}}, \quad 0 < \alpha < 1, \tag{2.39}$$

with

$$\hat{x} = \epsilon^{\alpha} x^*, \quad \tilde{x} = \epsilon^{\alpha-1} x^*.$$

Expanding the Helmholtz solutions (2.5) and (2.6) for $r_0^* = O(1)$, i.e. $\hat{r}_0 \ll 1$, and expanding the neck solutions (2.22), (2.29) and (2.30) for $r_0^* = O(1)$, i.e. $\tilde{r}_0 \gg 1$, one obtains

$$\left.\begin{aligned}
\hat{p}_0 &\approx \left\{1-\epsilon^{2\alpha}\frac{r_0^{*2}}{4}\right\}\left\{\hat{p}_i-\hat{A}_0\left[-\alpha\ln\frac{1}{\epsilon}+\ln\tfrac{1}{2}r_0^*+\gamma-\frac{i\pi}{2}\right]\right\}-\tfrac{1}{4}\epsilon^{2\alpha}\hat{A}_0 r_0^{*2},\\
\hat{p}_1 &\approx \cos\theta_0\left\{\epsilon^{-\alpha}\hat{A}_1 r_0^{*-1}-\epsilon^{\alpha}r_0^*\,i\hat{p}_i\sin\varphi\right.\\
&\qquad\left.-\epsilon^{\alpha}r_0^*\hat{A}_1\frac{1}{2}\left[-\alpha\ln\frac{1}{\epsilon}+\ln\tfrac{1}{2}r_0^*+\gamma-\tfrac{1}{2}-\frac{i\pi}{2}\right]\right\},
\end{aligned}\right\}\tag{2.40}$$

$$\left.\begin{aligned}
\tilde{p}_0 &\propto \{1-\tfrac{1}{4}\epsilon^{2}\epsilon^{2\alpha-2}r_0^{*2}\}\left\{\tilde{B}_0-\tilde{A}_0\left[(1-\alpha)\ln\frac{1}{\epsilon}+\ln r_0^*+D\right]\right\}-\tfrac{1}{4}\epsilon^{2\alpha}\tilde{A}_0 r_0^{*2},\\
\tilde{p}_1 &\propto \cos\theta_0\left\{\epsilon^{1-\alpha}\tilde{A}_1^{(+)}\frac{k}{Dr_0^*}+\epsilon^{\alpha-1}\tilde{A}_1^{(-)}\frac{D}{k}r_0^*\right.\\
&\qquad\left.-\epsilon^{\alpha+1}\tilde{A}_1^{(+)}\frac{r_0^* k}{2D}\left[(1-\alpha)\ln\frac{1}{\epsilon}+\ln r_0^*\right]\right\}.
\end{aligned}\right\}\tag{2.41}$$

In the overlap region $r_0^* = O(1)$ the expansions (2.40) and (2.41) have to be identical. This yields the following relations (independent of r_0^* and α) between the coefficients which are of course still functions of ϵ.

$$\left.\begin{aligned} \breve{A}_0(\epsilon) &= \hat{A}_0(\epsilon), \\ \breve{B}_0(\epsilon) &= \hat{p}_1 + \breve{A}_0(\epsilon)\left\{\ln\frac{1}{\epsilon} + D + \ln 2 - \gamma + \frac{i\pi}{2}\right\}, \\ \hat{A}_1(\epsilon) &= \epsilon \breve{A}_1^{(+)}(\epsilon)\,\frac{k}{D}, \\ \breve{A}_1^{(-)}(\epsilon) &= -i\epsilon\hat{p}_1\frac{k}{D}\sin\varphi + \epsilon^2\breve{A}_1^{(+)}(\epsilon)\frac{k^2}{2D^2}\left\{\ln\frac{1}{\epsilon} + \ln 2 - \gamma + \tfrac{1}{2} + \frac{i\pi}{2}\right\}. \end{aligned}\right\} \tag{2.42}$$

The same procedure is now applied to match the neck solutions to the cavity solutions (2.35) and (2.37). In general one has to choose an intermediate scaling between $\tilde{r}_1$ and $\bar{r}$, but, as pointed out earlier, the coordinate in the cavity solutions appears only in the combination $\delta\bar{r} \equiv \hat{r}_1$ such that we can use the same intermediate scaling (2.39) as before with $1 > \alpha > \ln\ln[1/\epsilon]/2\ln[1/\epsilon]$ instead of $1 > \alpha > 0$. Using now the expansions (2.36) and (2.38) of the cavity solutions for $r_1^* = O(1)$, i.e. $\bar{r} \ll 1$, we obtain

$$\left.\begin{aligned} \bar{p}_0 &\approx \bar{A}_0\{1 - \tfrac{1}{4}\epsilon^{2\alpha}r_1^{*2}\}\left\{1 - \alpha\frac{\mu_0\bar{R}^2}{2} + \frac{\mu_0\bar{R}^2}{2\ln[1/\epsilon]}(\ln\tfrac{1}{2}r_1^* + \gamma)\right\} + \epsilon^{2\alpha}\bar{A}_0\frac{\mu_0\bar{R}^2}{8\ln[1/\epsilon]}r_1^{*2}, \\ \bar{p}_1 &\approx \cos\theta_1\bar{A}_1\left\{\epsilon^{-\alpha}\frac{\mu_1\bar{R}^2}{2\ln[1/\epsilon]r_1^*} + \epsilon^{\alpha}\frac{r_1^*}{4}\left[2 + \alpha\mu_1\bar{R}^2 - \frac{\mu_1\bar{R}^2}{\ln[1/\epsilon]}(\ln\tfrac{1}{2}r_1^* + \gamma - \tfrac{1}{2})\right]\right\}, \end{aligned}\right\} \tag{2.43}$$

$$\left.\begin{aligned} \tilde{p}_0 &\propto \{1 - \tfrac{1}{4}\epsilon^{2\alpha}r_1^{*2}\}\left\{\breve{B}_0 + \breve{A}_0\left[(1-\alpha)\ln\frac{1}{\epsilon} + \ln r_1^* + D\right]\right\} \\ &\qquad + \tfrac{1}{4}\epsilon^{2\alpha}\breve{A}_0 r_1^{*2} - \epsilon^2\breve{B}_0\frac{4\bar{b}}{\pi}\left[(1-\alpha)\ln\frac{1}{\epsilon} + \ln r_1^*\right], \\ \tilde{p}_1 &\propto \cos\theta_1\left\{\epsilon^{1-\alpha}\breve{A}_1^{(-)}\frac{1}{Dr_1^*} - \epsilon^{\alpha+1}\breve{A}_1^{(-)}\frac{r_1^*}{2D}\left[(1-\alpha)\ln\frac{1}{\epsilon} + \ln r_1^*\right] + \epsilon^{\alpha-1}\breve{A}_1^{(+)}Dr_1^*\right\}. \end{aligned}\right\} \tag{2.44}$$

The identity between the two expansions (2.43) and (2.44) now leads to a second set of relations between the coefficients:

$$\left.\begin{aligned} \bar{A}_0(\epsilon)\frac{\mu_0\bar{R}^2}{2\ln[1/\epsilon]} &= \breve{A}_0(\epsilon) - \epsilon^2\frac{4\bar{b}}{\pi}\breve{B}_0(\epsilon), \\ \breve{B}_0(\epsilon)\left\{1 + \epsilon^2\ln\left[\frac{1}{\epsilon}\right]\frac{4\bar{b}}{\pi}\left[\frac{2}{\mu_0\bar{R}^2} - 1 + \frac{\gamma - \ln 2}{\ln[1/\epsilon]}\right]\right\} & \\ = \breve{A}_0(\epsilon)&\left\{\ln\left[\frac{1}{\epsilon}\right]\left(\frac{2}{\mu_0\bar{R}^2} - 1\right) + \gamma - \ln 2 - D\right\}, \\ \bar{A}_1(\epsilon) &= \epsilon\ln\left[\frac{1}{\epsilon}\right]\frac{2}{D\mu_1\bar{R}^2}\breve{A}_1^{(-)}(\epsilon), \\ \breve{A}_1^{(+)}(\epsilon) &= \epsilon^2\ln\left[\frac{1}{\epsilon}\right]\breve{A}_1^{(-)}(\epsilon)\frac{1}{D^2}\left\{\frac{1}{\mu_1\bar{R}^2} - \frac{2\gamma - 1 - 2\ln 2}{4\ln[1/\epsilon]}\right\}. \end{aligned}\right\} \tag{2.45}$$

Combining the relations (2.42) and (2.45) obtained from the outside and from the inside matching, one finally obtains for the source amplitude $\tilde{A}_0$

$$-\hat{p}_1 (2\tilde{A}_0)^{-1} = \ln\left[\frac{1}{\epsilon}\right]\left(1-\frac{1}{\mu_0 \bar{R}^2}\right)+D+\ln 2-\gamma+\tfrac{1}{4}\mathrm{i}\pi$$
$$+\epsilon^2\frac{2\tilde{b}}{\pi}\left\{\ln\left[\frac{1}{\epsilon}\right]\left(\frac{2}{\mu_0 \bar{R}^2}-1\right)+\gamma-\ln 2-D\right\}$$
$$\times\left\{\ln\left[\frac{1}{\epsilon}\right]\left(\frac{2}{\mu_0 \bar{R}^2}-1\right)+\gamma-\ln 2\right\}+O(\epsilon^4). \quad (2.46)$$

The above error estimate has to be interpreted as a block estimate, i.e. any powers of logarithms are allowed to multiply ϵ^4. Furthermore, we have for the dipole amplitude $\tilde{A}_1^{(-)}$

$$\tilde{A}_1^{(-)} = -\mathrm{i}\epsilon \hat{p}_i kD^{-1} \sin\varphi + O(\epsilon^3). \quad (2.47)$$

All other amplitudes are easily obtained from the relations (2.42) and (2.45). Using the results (2.25), (2.26) and (2.28) of §2.2, we can now determine the average $\tilde{x}$-velocity and pressure at the outside mouth plane $\tilde{x} = \tilde{b}$, which are needed for the impedance calculation, in the forms

$$\left.\begin{aligned}\bar{\tilde{u}} &= \frac{\mathrm{i}\pi}{2\epsilon}\tilde{A}_0+\epsilon c_u \tilde{A}_0,\\ \bar{\tilde{u}}^{-1} &= -\frac{2\mathrm{i}\epsilon}{\pi}\tilde{A}_0^{-1}+O(\epsilon^3\tilde{A}_0^{-1}),\\ \bar{\tilde{p}}_0 &= \tilde{B}_0-\tfrac{1}{2}\pi\tilde{b}\tilde{A}_0+O(\epsilon^2\tilde{A}_{01}, \epsilon^2\tilde{B}_0).\end{aligned}\right\} \quad (2.48)$$

If, for the calculation of the impedance, one takes the average mouth pressure as reference and uses (2.45) for the ratio $\tilde{B}_0/\tilde{A}_0$, one finds, as expected, a purely imaginary or reactive cavity impedance Z_c, as our model contains no dissipation mechanism whatsoever:

$$Z_c = -\frac{\bar{\tilde{p}}}{\bar{\tilde{u}}} = \mathrm{i}\epsilon\left\{\frac{2}{\pi}\left[\ln\left[\frac{1}{\epsilon}\right]\left(\frac{2}{\mu_0 \bar{R}^2}-1\right)+\gamma-\ln 2-D\right]-\tilde{b}+O(\epsilon^2)\right\}. \quad (2.49)$$

If, on the other hand, the pressure $\hat{p}_i$ on the baffle (in the absence of the resonator) is used as reference, one obtains the usual impedance Z_i, which now contains the radiation 'loss' of the cavity. Note that this 'loss' is not related to a dissipation mechanism, but is due to accounting only; it represents the amount of energy diverted from the incident plane wave into cylindrical waves emanating from the resonator mouth:

$$Z_i = -\frac{\hat{p}_i}{\bar{\tilde{u}}} = -\frac{4\mathrm{i}\epsilon}{\pi}\left\{\ln\left[\frac{1}{\epsilon}\right]\left(1-\underline{\frac{1}{\mu_0 \bar{R}^2}}\right)+D+\ln 2-\gamma+\tfrac{1}{4}\mathrm{i}\pi\right.$$
$$\left.+\epsilon^2\ln^2\left[\frac{1}{\epsilon}\right]\frac{2\tilde{b}}{\pi}\left(\underline{\frac{2}{\mu_0 \bar{R}^2}}-1\right)^2+O\left(\epsilon^2\ln\left[\frac{1}{\epsilon}\right]\right)\right\}. \quad (2.50)$$

The reason for keeping some $O(\epsilon^2)$ terms is to show that the volume in the naive spring-mass model has to include the neck volume as well. This is achieved by combining the two underlined terms in (2.50) to give

$$-\frac{\ln[1/\epsilon]}{\mu_0 \bar{R}^2}\left\{1-\epsilon^2\ln\left[\frac{1}{\epsilon}\right]\frac{8\tilde{b}}{\pi\mu_0 \bar{R}^2}\right\} \approx -\frac{\pi}{2}\left\{\frac{\pi\mu_0 \bar{R}^2}{2\ln[1/\epsilon]}+4\tilde{b}\epsilon^2\right\}^{-1}. \quad (2.51)$$

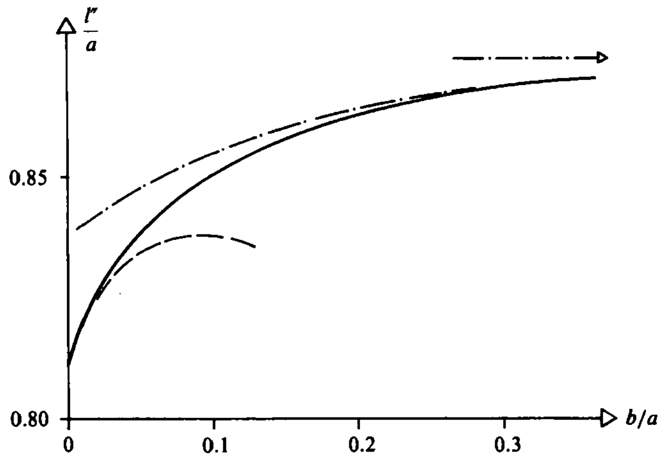

FIGURE 5. The neck-length-dependent part of the added length l''/a: ——, numerical evaluation of (2.53); — — —, approximation (2.54); —·—·—, approximation (2.55).

Apart from this, all $O(\epsilon^2)$ contributions will be dropped, as they do not contribute to physical insight. In the following, the comparison of (2.50) and (2.51) with the basic spring-mass model

$$Z_1 = \frac{\mathrm{i}S}{\ell V} - \mathrm{i}\ell\,[2b + l'] + \mathscr{R} \tag{2.52}$$

is carried out, where S is the mouth area ($S = 2a$ per unit length) and V the effective resonator volume per unit length, i.e. the geometric cavity volume V_c in the simplest model, and $\mathscr{R}$ the resistance, non-dimensionalized with $\rho_0 c_0$. Reverting to dimensional quantities, we find

$$\left.\begin{aligned} &V = \tfrac{1}{2}\pi R^2 \mu_0(\ell R) + 4ab, \quad \mathscr{R} = \ell a, \\ &\frac{l'}{a} = \frac{4}{\pi}\left\{\ln\frac{1}{\ell a} + \frac{l''}{a}\right\}, \quad \frac{l''}{a} = D - \frac{\pi b}{2a} + \ln 2 - \gamma. \end{aligned}\right\} \tag{2.53}$$

Using the expansions (2.23) and (2.24) for small and large b/a, we obtain the following approximations for l''/a, which are compared with the exact evaluation of (2.53) on figure 5:

for $b/a \ll 1$
$$\frac{l''}{a} \approx 2\ln 2 - \gamma + \frac{b}{a\pi}\left(\ln\frac{4\pi a}{b} + 1 - \frac{\pi^2}{2}\right) + \ldots, \tag{2.54}$$

for $b/a \gtrsim 0.15$
$$\frac{l''}{a} \propto \ln\tfrac{1}{2}\pi + 1 - \gamma - 2\exp\left[-2\left(\pi\frac{b}{a} + 2\right)\right] + \ldots. \tag{2.55}$$

At this point the results are in a form amenable to the following comments.

(*a*) It has been formally shown that the impedance does not depend on the angle of incidence φ of the plane wave.

(*b*) The effective volume in the spring term coincides with the physical cavity volume only in the limit $\epsilon \to 0$. For any finite ϵ the volume has to be corrected to account for the non-uniform pressure within the cavity. For the present semicylindrical geometry the exact form of this correction factor μ_0 has been derived. It is conjectured that for the lowest Helmholtz mode and other cavity shapes with geometric volume V_c and no dimension substantially different from $V_c^{\frac{1}{2}}$ the factor μ_0

will only be weakly affected if the equivalent radius R is defined as $R = (2V_c/\pi)^{\frac{1}{2}}$. Until now only heuristic corrections based typically on one-dimensional analysis have been used (see e.g. Dean 1974; Panton & Miller 1975). Another noteworthy feature is the fact that the neck volume has to be included in the effective volume. Although this is, in our formulation, a higher-order effect, it might improve predictions in cases where the neck volume is a sizeable fraction of the cavity volume.

(*c*) The comparison of the added length l'/a with results of Rayleigh (1945, 1904) and Morfey (1969) for the cases of constant pressure and constant velocity in the mouth plane shows that the leading contribution $(4/\pi)\ln[1/\epsilon]$ (for both sides) is recovered. It is easy to show that this contribution which is independent of b stems entirely from the wave field on both sides. So, for instance, it will drop to half the above value for a very small cavity with $\hat{R} = O(\epsilon)$. With some work it can be shown that for intermediate scalings $\hat{R} = O(\epsilon^{\beta} \times \text{arbitrary powers of logarithms})$ with $0 \leqslant \beta \leqslant 1$ the leading contribution becomes $2(2-\beta)/\pi \ln[1/\epsilon]$.

In addition we find that for $b = 0$ we recover the lower bound of $l''/a = 0.809079$, which is clear, as in this case the mouth plane is to leading order an equipotential surface. On the other hand l''/a does not reach Rayleigh's upper bound of $\frac{3}{2}-\gamma = 0.922785$. The expansion (2.55) shows that for long necks l''/a approaches the intermediate value 0.874368. It is recognized that the limit of long necks is, strictly speaking, incompatible with our assumption $b/a = O(1)$, but the limiting value is already practically attained for $b/a \approx 0.5$. For very long necks where the actual neck length is much larger than the added length, i.e. $b/a \gg \ln[1/\epsilon]$, the resonant balance (2.32) and the resulting scaling (2.33) would have to be modified. As it is of little practical use, this case will not be pursued.

(*d*) As expected, the resistance $\mathscr{R}$, to the order considered, is identical with the resistance of a strip piston.

For design purposes the accurate prediction of resonant frequencies is desirable, and has been somewhat cumbersome for two-dimensional geometries. Recalling that resonance is characterized by a minimum of $|Z_i|$, we find from (2.50), after neglecting all $O(\epsilon^2)$ terms, the resonance condition

$$\frac{1}{\mu_0 \bar{R}^2} = 1 + \frac{1}{\ln[1/\epsilon]}(\mathbb{D} + \ln 2 - \gamma), \tag{2.56}$$

which defines ω as a function of b/a and R. In the limit $\epsilon \rightarrow 0$, μ_0 is equal to unity and resonance occurs for $\bar{R} = 1$, which confirms our scaling argument in §2.3. At resonance Z_i and all amplitudes assume a particularly simple form:

$$\left.\begin{aligned} Z_i^{(\text{res})} &= \epsilon, \\ \tilde{A}_0^{(\text{res})} &= \hat{A}_0^{(\text{res})} = \frac{2i\hat{p}_i}{\pi} + O(\epsilon^2), \\ \tilde{B}_0^{(\text{res})} &= \frac{2i\hat{p}_i}{\pi}\left\{\ln\frac{1}{\epsilon} + \mathbb{D} + \ln 2 - \gamma\right\}, \\ \bar{A}_0^{(\text{res})} &= \frac{4i\hat{p}_i}{\pi\mu_0 \bar{R}^2}\ln\frac{1}{\epsilon}. \end{aligned}\right\} \tag{2.57}$$

These expressions allow the following conclusions.

(*a*) At resonance the amplitude of the radiated cylindrical wave is of the same order as the amplitude of the incident wave. Therefore extreme caution with the application of two-microphone methods to measure impedance is advisable in two-dimensional geometries.

(*b*) The pressure at the neck and that in the cavity are of order $\hat{p}_i \ln[1/\epsilon]$. From an inspection of (2.36) it appears that, while the uniform part of the cavity pressure is $\bar{A}_0$, the leading term which varies along $\bar{r}$ is of order $\bar{A}_0/\ln[1/\epsilon]$, i.e. not very much smaller than the uniform part for most applications. This makes the appearance of the volume-correction factor physically plausible. It is noted that, for the semicylindrical cavity geometry considered here, the results are also applicable for $\hat{R} = O(1)$, where the zeros and poles of μ_0 are of particular interest. At a zero of μ_0 or, in other words, at a resonance of the cavity alone, the impedance is infinite and the velocity at the mouth becomes zero. This is consistent with our linearized inviscid approach, which prohibits feeding of energy into a resonating cavity. At a pole of μ_0, on the other hand, the cavity appears like an infinite half-space. The impedance (2.50) becomes identical with the impedance of a slotted baffle without attached cavity with the only exception of the resistance: the resistance for the slotted baffle is twice the value given by (2.50), as the equivalent 'piston' at the slot radiates into both half-spaces.

In addition, we also find higher Helmholtz modes with resonance frequencies given by (cf. 2.56)

$$\ln\left[\frac{1}{\epsilon}\right](\mu_0(\hat{R})\,\hat{R}^2)^{-1} \approx 1. \tag{2.58}$$

This implies that these higher modes correspond to radii $\hat{R}$ that are smaller than the poles of μ_0 by an amount of order $(\ln[1/\epsilon])^{-1}$. This finding is in agreement with Miles' (1971) estimate in his equation (1.6); for a simplified one-dimensional approach see also Panton & Miller (1975).

(*c*) Finally we demonstrate that the resonance condition (2.56) yields useful results by comparing with an experiment of Walker & Charwat (1982). A geometry of (equivalent) $R = 234$ mm, $a = 3.81$ mm and $b = 3$ mm resulted in a resonance frequency of 100 Hz. Using $c_0 = 340$ m/s, we find $\epsilon = 0.007$ and, with (2.24), (2.56) and figure 4, $\mu_0 = 0.875$ and a computed radius $R = 216$ mm, which is only 7.7 % in error. Considering the fact that the span of the orifice was only 150 mm, making the radiation field three-dimensional, this agreement is as good as can be expected.

(*d*) Finally, we note that as $\epsilon \rightarrow 0$ the velocity in the neck blows up like $\hat{p}_i/\epsilon$, i.e. for a given incident pressure and a narrowing neck the velocity increases in inverse proportion to the available flow area, which is physically sensible. The point to be made here is that for the linearized treatment to be valid the ratio $\hat{p}_i/\epsilon$ has to remain small as $\epsilon \rightarrow 0$. Otherwise the limit process $\epsilon \rightarrow 0$ is physically meaningless.

In §2.5 we close the discussion of the two-dimensional resonator by considering its energy budget.

2.5. *The energy balance*

In this subsection it is shown explicitly that the time-averaged acoustic power fed into the resonator is equal to the reflected plus the radiated power. This serves as a consistency check as no dissipation has been included in the analysis. The time-averaged energy-flux vector is given by $\frac{1}{2}\,\mathrm{Re}\,(p\boldsymbol{u}^*)$, where the asterisk denotes the complex conjugate. To compute the radial energy flux, the pressure in the Helmholtz region given by (2.5) and (2.6) and the radial velocity $v = -\mathrm{i}\,\partial p/\partial r$ associated with it are as follows (the dipole radiation, being of order $\epsilon^4 \ln[1/\epsilon]$ (cf. 2.47), has been neglected):

$$\left.\begin{aligned}\hat{p} &= \hat{p}_i \cos[\hat{r}_0 \cos\varphi \sin\theta_0]\exp[-\mathrm{i}\hat{r}_0 \sin\varphi\cos\theta_0] + \tfrac{1}{2}\mathrm{i}\pi\hat{A}_0\, H_0^{(1)}(\hat{r}_0),\\ \hat{v} &= \hat{p}_i \exp[-\mathrm{i}\hat{r}_0 \sin\varphi\cos\theta_0]\\ &\quad\times\{-\sin\varphi\cos\theta_0\cos[\hat{r}_0\cos\varphi\sin\theta_0]\\ &\quad+\mathrm{i}\,\cos\varphi\sin\theta_0\sin[\hat{r}_0\cos\varphi\sin\theta_0]\} - \tfrac{1}{2}\pi\hat{A}_0\, H_1^{(1)}(\hat{r}_0).\end{aligned}\right\} \tag{2.59}$$

The radial energy flux is now integrated over a shell of radius $\hat{r}_0$. To obtain additional insights, this is done for the fourth and first quadrant separately, which will allow study of the 'transmission' of energy between these quadrants when the plane-wave incidence is not normal. After a fair amount of rather straightforward work, one obtains for the power transmitted through the shells of radius $\hat{r}_0$ in the fourth (IV) and first (I) quadrants (assuming $\hat{p}_i$ real)

$$\Pi_{\mathrm{I}}^{\mathrm{IV}} = \mp \hat{p}_i^2 \frac{\hat{r}_0 \sin\varphi}{4}\left\{\frac{\sin[2\hat{r}_0\cos\varphi]}{2\hat{r}_0\cos\varphi}+1\right\}+\frac{1}{2}\left(\frac{\pi}{2}\right)^2\left\{|\hat{A}_0|^2-\frac{2}{\pi}\hat{p}_i\,\mathrm{Im}\,(\hat{A}_0)\right\}$$
$$\pm\tfrac{1}{4}\pi\hat{r}_0\hat{p}_i\left\{[\mathrm{Re}(\hat{A}_0)\,Y_1(\hat{r}_0)+\mathrm{Im}\,(\hat{A}_0)\,J_1(\hat{r}_0)]\int_0^{\varphi}\sin[\hat{r}_0\cos z]\,\mathrm{d}z\right.$$
$$\left.+[\mathrm{Re}(\hat{A}_0)\,Y_0(\hat{r}_0)+\mathrm{Im}\,(\hat{A}_0)\,J_0(\hat{r}_0)]\int_0^{\varphi}\cos z\cos[\hat{r}_0\cos z]\,\mathrm{d}z\right\}. \qquad (2.60)$$

Considering the expression (2.46) for $\tilde{A}_0 = \hat{A}_0$, it is immediately shown that the second term in (2.60) above is identically zero. Addition of Π_{IV} and Π_{I} then yields zero, as required by our approach.

In addition it is now possible to evaluate the modification of the first term in (2.60), which corresponds to the plane wave and its reflection alone, by the resonator. This modification, the third term in (2.60), can be simplified for grazing incidence $\varphi = \frac{1}{2}\pi$. The integrals become Struve functions (cf. Abramowitz & Stegun 1968), which can be further simplified by considering a large 'control-shell' radius $\hat{r}_0$:

$$\Pi_{\mathrm{I}}^{\mathrm{IV}}(\varphi=\tfrac{1}{2}\pi) = \mp\tfrac{1}{2}\hat{p}_i^2\hat{r}_0 \pm \tfrac{1}{2}(\tfrac{1}{2}\pi)^2\hat{p}_i\hat{r}_0\left\{[\mathrm{Re}\,(\hat{A}_0)\,Y_1(\hat{r}_0)+\mathrm{Im}\,(\hat{A}_0)\,J_1(\hat{r}_0)]\,\boldsymbol{H}_0(\hat{r}_0)\right.$$
$$\left.+[\mathrm{Re}\,(\hat{A}_0)\,Y_0(\hat{r}_0)+\mathrm{Im}\,(\hat{A}_0)\,J_0(\hat{r}_0)]\left[\frac{2}{\pi}-\boldsymbol{H}_1(\hat{r}_0)\right]\right\}$$
$$\propto \mp\frac{\hat{p}_i^2\hat{r}_0}{2}\pm\left(\frac{\pi}{2}\right)^2\frac{\hat{p}_i\,\mathrm{Im}\,(\hat{A}_0)}{\pi} \quad (\hat{r}_0 \gg 1). \qquad (2.61)$$

As the imaginary part of $\hat{A}_0$ is positive, it follows that, depending on the incidence angle of the forcing plane wave, a part of the incident energy is returned to the sender instead of being transmitted into the first quadrant. From (2.61) it is also clear that this phenomenon is associated with the radiation resistance, as it is the imaginary part of $\hat{A}_0$ that determines the resistance in (2.50). Therefore, when designing a sound-absorbing wall with the optimum amount of resistance, the radiation resistance should not be added to the dissipative resistance for normal sound incidence, while it can be taken into account using (2.60) for other incidence angles.

3. The three-dimensional resonator

In this section we consider a three-dimensional Helmholtz resonator consisting of a circular hole of radius a in an infinitely thin baffle and a hemispherical cavity. The geometry is shown in figure 6, where the x-axis is taken as the symmetry axis.

The analysis presented below has in part been suggested by Pierce (1981), and will only be sketched as all the basic ideas have been developed in §2.

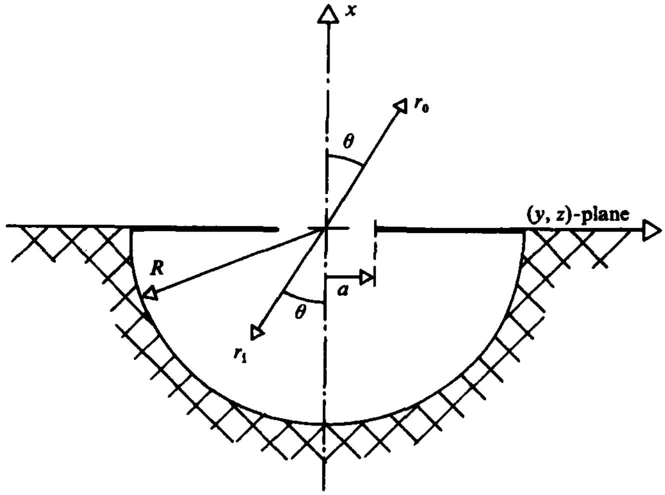

FIGURE 6. The geometry of the 3-dimensional Helmholtz resonator.

3.1. *The Helmholtz region*

To simplify matters, we prescribe an excitation by a plane wave running in the negative $\hat{x}$-direction, i.e. at normal incidence only. In analogy to (2.5), the incoming wave is combined with the wave reflected from the baffle at $\hat{x} = 0$ to

$$\hat{p} = \hat{p}_i \cos \hat{x},$$

$$\hat{p} \approx \hat{p}_i \{1 - \tfrac{1}{12}\hat{r}_0^2 [2 + (3\cos 2\theta + 1)] + \ldots\} \quad (\hat{r}_0 \to 0). \tag{3.1}$$

In the small-$\hat{r}_0$ expansion above, the radially symmetric terms have been separated from terms proportional to higher zonal harmonics like $3\cos 2\theta + 1$ etc. As in the two-dimensional case, we will concentrate on the zeroth-order zonal harmonic or radially symmetric solution. The spherical wave analogous to (2.6) emanating from the orifice is of the form

$$\hat{p}_0 = \hat{A}_0 \frac{e^{i\hat{r}_0}}{\hat{r}_0},$$

$$\hat{p}_0 \approx \hat{A}_0 \hat{r}_0^{-1} \{1 + i\hat{r}_0 - \tfrac{1}{2}\hat{r}_0^2 + \ldots\}. \tag{3.2}$$

3.2. *The orifice region*

The coordinates in this region are again rescaled according to (2.8) and (2.9), which leaves us with the task of solving the Laplace equation in the orifice geometry. This problem of incompressible ideal flow through a circular orifice has been solved by Lamb (1945, art. 107). Denoting the polar angle in the (y, z)-plane by φ, the appropriate coordinate system is

$$\tilde{x} = \cos\vartheta \sinh\eta, \quad \tilde{y} = w\cos\varphi, \quad \tilde{z} = w\sin\varphi, \quad w \equiv \sin\vartheta\cosh\eta. \tag{3.3}$$

The coordinate η ranges from $-\infty$ to $+\infty$ and ϑ ranges from zero to $+\frac{1}{2}\pi$. The potentials satisfying Laplace's equations are given by Lamb as

$$\phi_0 = \cot^{-1}\zeta, \tag{3.4}$$

$$\phi_2 = 2(3\cos^2\vartheta - 1)\,[(3\zeta^2 + 1)\cot^{-1}\zeta - 3\zeta], \tag{3.5}$$

$$\vdots$$

$$\zeta \equiv \sinh\eta.$$

Noting that $\zeta^2 = \tilde{r}^2 - \sin^2\vartheta$ and considering $\sin^2\vartheta = (\tilde{y}^2+\tilde{z}^2)/\tilde{r}^2$ to obtain a relation between θ and ϑ, we obtain for large $\tilde{r}$

$$\zeta \propto \tilde{r}\left\{1-\frac{\sin^2\theta}{2\tilde{r}^2}+\frac{\sin^2\theta(4-5\sin^2\theta)}{8\tilde{r}^4}+\ldots\right\}. \tag{3.6}$$

$$\left.\begin{aligned}\cot^{-1}\zeta &\propto \frac{1}{\tilde{r}_0}-\frac{3\cos 2\theta+1}{12\tilde{r}_0^3}+\frac{35\sin^4\theta-40\sin^2\theta+8}{40\tilde{r}_0^5}+\ldots \quad (\tilde{r}\to\infty)\\ \cot^{-1}\zeta &\propto \pi-\frac{1}{\tilde{r}_\mathrm{i}}+\frac{3\cos 2\theta+1}{12\tilde{r}_\mathrm{i}^3}+\ldots \quad (\tilde{r}_\mathrm{i}\to\infty).\end{aligned}\right\} \tag{3.7}$$

This provides directly the asymptotic expansions for ϕ_0. The trigonometric expressions above thereby represent zonal harmonics. As in the two-dimensional case it can be shown that the omission of higher zonal harmonics does not affect the impedance to the order considered in §3.4. For this reason they will be omitted in the following.

By subtracting the value of ϕ_0 at the orifice, i.e., $\eta = 0$ we can write

$$\left.\begin{aligned}\tilde{p}_{0,0} &\propto \tilde{A}_0\left\{\frac{1}{\tilde{r}_0}-\frac{\pi}{2}\right\}+\tilde{B}_0 \quad (\tilde{r}_0\to\infty)\\ \tilde{p}_{0,0} &\propto -\tilde{A}_0\left\{\frac{1}{\tilde{r}_\mathrm{i}}-\frac{\pi}{2}\right\}+\tilde{B}_0 \quad (\tilde{r}_\mathrm{i}\to\infty)\\ \tilde{\bar{p}}_{0,0} &= \tilde{B}_0; \quad \tilde{\bar{u}}_{0,0} = 2\mathrm{i}\tilde{A}_0.\end{aligned}\right\} \tag{3.8}$$

The value for the average velocity is obtained directly from (3.3) to (3.5) at small η or from the application of the divergence theorem. Finally we also need the particular solution of $\tilde{\nabla}^2\tilde{p}_{0,1} = -\tilde{p}_{0,0}$ which is found to be

$$\left.\begin{aligned}\tilde{p}_{0,1} &\propto -\tilde{A}_0\left\{\frac{\tilde{r}_0}{2}-\frac{\pi\tilde{r}_0^2}{12}\right\}-\tilde{B}_0\frac{\tilde{r}_0^2}{6} \quad (\tilde{r}_0\to\infty)\\ \tilde{p}_{0,1} &\propto +\tilde{A}_0\left\{\frac{\tilde{r}_\mathrm{i}}{2}-\frac{\pi\tilde{r}_\mathrm{i}^2}{12}\right\}-\tilde{B}_0\frac{\tilde{r}_\mathrm{i}^2}{6} \quad (\tilde{r}_\mathrm{i}\to\infty).\end{aligned}\right\} \tag{3.9}$$

3.3. *The cavity*

The same scaling argument as for the two-dimensional case in §2.3 applies here. The estimate (2.32) with $l' = O(a)$ yields

$$\delta = \epsilon^{\frac{1}{3}}, \quad \bar{R} = \hat{R}\epsilon^{-\frac{1}{3}}. \tag{3.10}$$

For the hemispherical cavity under consideration, the full Helmholtz equation (2.3) can again be solved subject to the impermeability condition at $\hat{r}_\mathrm{i} = \hat{R}$, with the result

$$\bar{p}_0 = \frac{\bar{A}_0}{\hat{r}_\mathrm{i}}\{-\tfrac{1}{3}\epsilon\bar{R}^3\mu_0(\epsilon^{\frac{1}{3}}\bar{R})\cos\hat{r}_\mathrm{i}+\sin\hat{r}_\mathrm{i}\}, \tag{3.11}$$

$$\mu_0(z) = 3z^{-3}\tan[z-\tan^{-1}z],$$

$$\bar{p}_0 \approx \frac{\bar{A}_0}{\hat{r}_\mathrm{i}}\{-\tfrac{1}{3}\epsilon\bar{R}^3\mu_0[1-\tfrac{1}{2}\hat{r}_\mathrm{i}^2]+\hat{r}_\mathrm{i}-\tfrac{1}{6}\hat{r}_\mathrm{i}^3+O(\hat{r}_\mathrm{i}^4)\} \quad (\hat{r}_\mathrm{i}\to 0).$$

In analogy to the two-dimensional case, the correction factor $\mu_0(\epsilon^{\frac{1}{3}}\bar{R})$ has been normalized to 1 in the limit of vanishing argument, and has the Taylor expansion

$$\mu_0(z) \approx 1-\tfrac{3}{5}z^2+O(z^4). \tag{3.12}$$

3.4. *Matching and results*

In the following the formal matching procedure is omitted, as it is straightforward. Using $\mu_0 = O(1)$ and, in this case, either the intermediate matching procedure of Cole, or Van Dyke's matching principle, one arrives at the amplitudes

$$\bar{A}_0 = \frac{3}{\mu_0 \bar{R}^3}\tilde{A}_0, \quad \hat{A}_0 = \epsilon \tilde{A}_0, \quad \tilde{B}_0 = \left\{\frac{3}{\mu_0 \bar{R}^3} - \frac{\pi}{2}\right\}\tilde{A}_0, \quad \tilde{A}_0\left\{\frac{3}{\mu_0 \bar{R}^3} - \pi - \mathrm{i}\epsilon\right\} = \hat{p}_1. \quad (3.13)$$

With the result (3.8) for $\bar{\bar{u}}_{0,0}$, and noting that $\bar{\bar{u}}_{0,1}$ (cf. (2.11)) is proportional to $\tilde{A}_0$ (with an unknown proportionality constant), we obtain, for the impedance based on the incident pressure $\hat{p}_1$,

$$Z_1 = -\frac{\hat{p}_1}{\bar{\bar{u}}} = \mathrm{i}\epsilon\left\{\frac{3}{2\bar{R}^3 \mu_0(\hat{R})} - \frac{\pi}{2} - \mathrm{i}\frac{\epsilon}{2} + O(\epsilon^2)\right\}. \quad (3.14)$$

Comparison of this result with the spring-mass model (2.52) yields

$$V = \frac{2\pi}{3}R^3 \mu_0(kR), \quad \mathscr{R} = \tfrac{1}{2}(ka)^2, \quad \frac{l'}{a} = \tfrac{1}{2}\pi. \quad (3.15)$$

Once again, the basic ideas of the spring-mass model are confirmed. In addition, the exact volume-correction factor for a hemispherical cavity has been derived. The added length is twice (for both sides) Rayleigh's (1945) result for a constant-pressure 'piston', as the orifice plane, according to (3.4), is an equipotential surface. For non-zero baffle thickness this added length is expected to increase to a value bounded from above by $16a/3\pi$, twice the result obtained for a circular piston in a baffle (Rayleigh 1945, Appendix A).

The resistance is again identical with the resistance of a circular piston radiating into a half-space.

Finally we find, in analogy to (2.56), the resonance condition

$$\frac{\bar{R}^3}{3}\mu_0(\hat{R}) = \frac{1}{\pi}. \quad (3.16)$$

This condition has to be compared with Panton & Miller's (1975) equation (4): the two conditions are identical in the limit of vanishing ϵ. The first correction term obtained from expanding μ_0 (cf. 3.12), on the other hand, has the opposite sign to that of Panton & Miller, who consider a 'one-dimensional' elongated cavity. This serves as an illustration that neither result should be applied to case where the cavity has radically different proportions.

From (3.13) the amplitudes at resonance are easily computed as

$$\tilde{A}_0^{(\mathrm{res})} = \mathrm{i}\epsilon^{-1}\hat{p}_1, \quad \hat{A}_0^{(\mathrm{res})} = \mathrm{i}\hat{p}_1, \quad \bar{A}_0^{(\mathrm{res})} = \pi \tilde{A}_0^{(\mathrm{res})}. \quad (3.17)$$

Again, at resonance the amplitude $\hat{A}_0$ of the radiated spherical wave is of the same order as the incident pressure.

4. Conclusions

The impedance of a two- and three-dimensional resonator has been rigorously determined in the limit of long acoustic wavelength and small amplitude by using the method of matched asymptotic expansions.

The results for a two-dimensional resonator indicate that Rayleigh's upper bound for the added length (for uniform velocity at the mouth) is not attained. A refined upper bound is given, while it is shown that the leading-order contribution to the added length is independent of the neck geometry and is determined by the cavity size and shape. In both cases (two and three dimensions) the effective cavity volume to be used in the simple spring-mass model is determined. It differs from the geometric volume by a correction factor, which takes the pressure variation in the cavity into account. In addition it is shown in the two-dimensional case that the neck volume has to be added to the effective cavity volume.

Finally, it is shown explicitly that the 'radiation-resistance' term in the impedance is related to a redistribution of energy and not to a loss. Its influence on the reflection coefficient depends on the incidence angle of the forcing plane wave. 'Dissipative resistance', on the other hand, is related to viscous, nonlinear and grazing-flow effects, which are all beyond the scope of this investigation. Models for the effect of nonlinearity (jetting) can be found in Ingard & Ising (1967), van Wijngaarden (1968) and Tang & Sirignano (1973), for instance, while the effect of grazing flow has been modelled by Ronneberger (1972), Howe (1979) and Walker & Charwat (1984), among others.

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